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Integral Curves and Flc-Frames: A Comprehensive Study in Timelike Minkowski 3-Space

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Abstract

This work generalizes the Frenet-like frame to timelike curves in Minkowski space, providing a unified framework for cases where the Frenet frame fails, particularly when the second derivative of a curve vanishes. We derive the Flc-frame and Flc-equations for timelike curves, introducing the Flc-curvatures K_1, K_2 and K_3, and establish explicit relationships between them. A key contribution is the formulation of new integral curves generated from the Flc-frame, which allows for the explicit computation of Frenet vectors, curvature, and torsion in terms of the Flc-curvatures. We provide a detailed example to illustrate the application of the Flc-frame, demonstrating its effectiveness in analyzing the geometric properties of poly-timelike curves. Additionally, we explore the relationship between the Flc-curvatures and derive integral curves based on the Flc-normal and Flc-binormal vectors. These results extend the understanding of curve analysis in Minkowski 3-space and offer a robust framework for studying polynomial curves, particularly in scenarios where the Frenet frame fails.

Keywords: Minkowski 3-space; Flc-frame; Flc-equations; Flc-curvatures; polynomial curve; integral curves.

1 Introduction

In Minkowski 3-space $\mathbb{E}_1^3$, the study of curves is a central topic in differential geometry and mathematical physics. The traditional Frenet frame, consisting of the tangent, normal, and binormal vectors, is a fundamental tool for analyzing the geometric properties of curves. However, as demonstrated by Bishop [1], the Frenet frame fails at points where the second derivative vanishes.

To address this issue, alternative frames in Minkowski space have been introduced, such as the Bishop frame [15] and the equiform frame [4]. These frames provide more robust tools for curve analysis, especially in cases where the Frenet frame fails. Also in Euclidean space, some frames were introduced and studied, such as the quasi frame [7] and the modified frame [9].

Among these, the Frenet-like curve (Flc) frame [3, 6] is a powerful tool for studying polynomial curves in Euclidean 3-space. The Flc-frame defines the binormal vector as the cross product of the first and n-th derivatives of the curve, offering a geometric interpretation of higher-order derivatives [8, 14]. Compared to the Frenet and Bishop frames, the Flc-frame simplifies calculations and reduces the number of singular points, making it particularly suitable for analyzing polynomial curves [13].

In this paper, we extend the application of the Flc-frame to timelike polynomial (poly-timelike) curves in Minkowski 3-space. We derive the Flc-frame and Flc-equations for timelike curves, introducing the Flc-curvatures K_1, K_2 and K_3, and establish explicit relationships between them. These curvatures provide a comprehensive description of the geometric properties of timelike curves, even in cases where the Frenet frame fails. Additionally, we present a detailed example to illustrate the application of the Flc-frame, demonstrating its effectiveness in analyzing the geometric properties of poly-timelike curves.

A key contribution of this work is the formulation of new integral curves generated from the Flc-frame. These integral curves are defined by integrating the Flc-normal and Flc-binormal vectors, and we derive their Frenet vectors, curvature, and torsion in terms of the Flc-curvatures. This approach extends the understanding of curve analysis in Minkowski 3-space and provides a robust framework for studying polynomial curves, particularly in scenarios where the Frenet frame fails. The results highlight the advantages of the Flc-frame over traditional methods, offering a versatile tool for both theoretical and practical applications in differential geometry and mathematical physics.

The paper is organized as follows: In Section 2, we review the preliminaries, including the definition of Minkowski 3-space and the classification of curves based on their causal character. Section 3 focuses on the Flc-frame and Flc-equations for poly-timelike curves, providing explicit expressions for the Flc-curvatures and their relationships. In Section 4, we introduce new integral curves and derive their Frenet apparatus in terms of the Flc-frame and Flc-curvatures. Finally, we conclude with a discussion of the implications of our results and potential applications.

This work not only advances the theoretical understanding of polynomial curves in Minkowski space but also provides practical tools for analyzing curves in relativistic and geometric contexts. By generalizing the Frenet frame and introducing new integral curves, we offer a comprehensive framework that can be applied to a wide range of problems in differential geometry and mathematical physics.

2 Preliminaries

In this section, we establish the foundational concepts and notations necessary for understanding the results presented in this paper. We begin by defining the Lorentz-Minkowski space and its metric structure, followed by a discussion of the causal character of vectors and curves. Finally, we review the Frenet frame and its limitations, setting the stage for the introduction of the Flc-frame.

2.1 Lorentz-Minkowski space

The Lorentz-Minkowski space, denoted by $\mathbb{E}_1^3$, is a 3-dimensional space equipped with a metric tensor g that distinguishes it from Euclidean space. Formally, $\mathbb{E}_1^3$ is the pair (E^3, g), where E^3 represents the underlying Euclidean 3-space, and the metric g is given by,

$$g = -du_1^2 + du_2^2 + du_3^2,$$

where (u_1, u_2, u_3) is a coordinate system in $\mathbb{E}_1^3$. This metric induces a pseudo-Riemannian structure, which is central to the study of curves and surfaces in Minkowski space [10, 16].

2.2 Causal character of vectors and curves

In $\mathbb{E}_1^3$, the causal character of a vector M is determined by the sign of its inner product with itself under the metric g. Specifically:

- A vector M is *spacelike* (*Sp-like*) if $g(M, M) > 0$ or $M = 0$.
- A vector M is *timelike* (*T-like*) if $g(M, M) < 0$.
- A vector M is *lightlike* (*null*) if $g(M, M) = 0$ and $M \neq 0$.

For a vector M, the norm is defined as $\|M\| = \sqrt{|g(M, M)|}$. If $\|M\| = 1$, the vector is said to be unit. Two vectors M_1 and M_2 are orthogonal if $g(M_1, M_2) = 0$ [4].

The vector product (cross product) of two vectors $M_1 = (a_1, a_2, a_3)$ and $M_2 = (h_1, h_2, h_3)$ in $\mathbb{E}_1^3$ is defined as:

$$M_1 \wedge M_2 = (a_3h_2 - a_2h_3,\ a_3h_1 - a_1h_3,\ a_1h_2 - a_2h_1).$$

2.3 Curves in Minkowski space

A curve in $\mathbb{E}_1^3$ is a smooth mapping $\omega(s) : I \to \mathbb{E}_1^3$, where I is an open interval in $\mathbb{R}$, and s is the arc-length parameter [2, 12]. The causal character of a curve $\omega(s)$ is determined by its tangent vector $\dfrac{d\omega(s)}{ds}$:

- $\omega(s)$ is *spacelike* if $\dfrac{d\omega(s)}{ds}$ is spacelike for all $s \in I$.

- $\omega(s)$ is *timelike* if $\dfrac{d\omega(s)}{ds}$ is timelike for all $s \in I$.
- $\omega(s)$ is *lightlike* if $\dfrac{d\omega(s)}{ds}$ is lightlike for all $s \in I$.

For non-null curves, the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is commonly used, where

- $\mathbf{T}(s)$ is the unit tangent vector.
- $\mathbf{N}(s)$ is the principal normal vector.
- $\mathbf{B}(s)$ is the binormal vector [11].

The Frenet equations for a timelike curve $\omega(s)$ with a spacelike normal $\mathbf{N}$ are given by:

$$\frac{d}{ds}\begin{bmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{bmatrix} = \begin{bmatrix}0 & \kappa(s) & 0\\ \kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{bmatrix}, \tag{1}$$

where $\kappa(s)$ and $\tau(s)$ are the curvature and torsion of the curve, respectively. The inner products of the Frenet vectors satisfy,

$$g(\mathbf{T}(s), \mathbf{T}(s)) = -1, \qquad g(\mathbf{N}(s), \mathbf{N}(s)) = 1, \qquad g(\mathbf{B}(s), \mathbf{B}(s)) = 1,$$

and

$$\mathbf{T}(s) \times \mathbf{N}(s) = \mathbf{B}(s), \qquad \mathbf{N}(s) \times \mathbf{B}(s) = \mathbf{T}(s), \qquad \mathbf{B}(s) \times \mathbf{T}(s) = -\mathbf{N}(s).$$

When the curve is not parameterized by arc length, the Frenet frame is given by:

$$\mathbf{T}(t) = \frac{\omega'(t)}{\|\omega'(t)\|}, \qquad \mathbf{N}(t) = -\mu\mathbf{T}(t) \times \mathbf{B}(t), \qquad \mathbf{B}(t) = \frac{\omega'(t) \times \omega''(t)}{\|\omega'(t) \times \omega''(t)\|}, \tag{2}$$

where $\mu = -1$ if $\mathbf{N}(t)$ is spacelike and $\mu = 1$ if $\mathbf{N}(t)$ is timelike. The curvature κ and torsion τ are given as,

$$\kappa = \frac{\|\omega'(t) \times \omega''(t)\|}{\|\omega'(t)\|^3}, \qquad \tau = \frac{g(\omega'''(t), \omega'(t) \times \omega''(t))}{\|\omega'(t) \times \omega''(t)\|^2}. \tag{3}$$

2.4 Limitations of the Frenet frame

While the Frenet frame is widely used, it fails at points where the curvature κ vanishes, as the normal and binormal vectors become undefined, so the Flc-frame has been studied in more papers such as [6, 14]. This limitation motivates the need for alternative frames, such as the Flc-frame, which we introduce and analyze in this work. The Flc-frame provides a more robust framework for studying polynomial curves, particularly in Minkowski space, where the causal character of vectors and curves adds additional complexity.

3 Timelike Polynomial Curves

In this section, the timelike Flc-frame and timelike Flc-equations in $\mathbb{E}_1^3$ are investigated. In addition, the Flc-curvatures of the curve, K_1, K_2 and K_3 are introduced. Moreover, a relationship between the Flc-curvatures is obtained.

3.1 Flc-frame and Flc-equations of timelike curves in $\mathbb{E}_1^3$

In this subsection, we study the timelike curves with a spacelike Flc-normal vector, considering the curve is a poly-spatial curve of degree n and the n-th derivative is either a spacelike or a timelike vector in $\mathbb{E}_1^3$.

Definition 3.1. *If $\omega(t)$ is a poly-timelike curve of degree n, then the Flc-binormal vector $D_1(t)$ is the unique spacelike unit vector perpendicular to the timelike plane $\{T(t), D_2(t)\}$, and we can define the Flc-frame consisting of $\{T(t), D_2(t), D_1(t)\}$ the tangent, Flc-normal, Flc-binormal vectors, respectively, by:*

$$T(t) = \frac{\omega'(t)}{\parallel \omega'(t) \parallel}, \qquad D_1(t) = \frac{\omega'(t) \times \omega^{(n)}(t))}{\parallel \omega'(t) \times \omega^{(n)}(t) \parallel}, \qquad D_2(t) = D_1(t) \times T(t). \tag{4}$$

Theorem 3.1. *If $\omega(t)$ is a poly-timelike curve of degree n, then the Flc-equations can be formulated as follows:*

$$\frac{d}{dt}\begin{bmatrix} T(t) \\ D_2(t) \\ D_1(t) \end{bmatrix} = u \begin{bmatrix} 0 & K_1 & K_2 \\ K_1 & 0 & K_3 \\ K_2 & -K_3 & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ D_2(t) \\ D_1(t) \end{bmatrix}, \tag{5}$$

where $u = \parallel \omega'(t) \parallel = \dfrac{ds}{dt}$.

Proof. Define the Lorentzian angle ψ between the Frenet normal vector N and the Flc-normal vector D_2 in which:

$$D_2 = \cos\psi\ N + \sin\psi\ B, \tag{6}$$

$$D_1 = -\sin\psi\ N + \cos\psi\ B. \tag{7}$$

By using (1), we can obtain,

$$T'(t) = u\ \kappa\ N = \ u\kappa\cos\psi D_2 - u\kappa\sin\psi D_1.$$

Differentiating (6), and substituting (5) and (7), we get

$$D_2'(t) = u(\kappa\ \cos\psi)T(t) + u\left(\frac{d\psi}{ds} + \tau\right) D_1(t).$$

Similarly, differentiating (7), and substituting (5) and (6), we have

$$D_1'(t) = -u(\kappa\ \sin\psi)T(t) - u\left(\frac{d\psi}{ds} + \tau\right) D_2(t).$$

By taking the substitution,

$$K_1 = \kappa\cos\psi,$$
$$K_2 = -\kappa\sin\psi,$$
$$K_3 = \frac{d\psi}{ds} + \tau.$$

The results are obtained. □

Corollary 3.1. *If $\omega(t)$ is a poly-timelike curve of degree n, and if $\omega^{(n)}$ is a spacelike vector, i.e., D_1 is a spacelike vector. Then, the new curvatures K_1, K_2, and K_3 are respectively defined by:*

$$\begin{aligned} K_1 &= \frac{g(T', D_2)}{u} = -\frac{g(D_2', T)}{u}, \\ K_2 &= \frac{g(T', D_1)}{u} = -\frac{g(D_1', T)}{u}, \\ K_3 &= \frac{g(D_2', D_1)}{u} = -\frac{g(D_1', D_2)}{u}. \end{aligned} \tag{8}$$

3.2 The relation between the Flc-curvatures

In this subsection, we give equations for determining the Flc-curvatures, $K_1,\ K_2$ and K_3, and introduce the relation between them in the case of timelike curves.

Theorem 3.2. *If $\omega(t)$ is a poly-timelike curve of degree n, and if the $n-th$ derivative $\omega^{(n)}(t)$ lies in the same space. The curvatures $K_1, K_2$and K_3 of the curve can be given by:*

$$K_1 = \frac{g(\omega'(t) \times \omega''(t),\ \omega'(t) \times \omega^{(n)}(t)}{\| \omega'(t) \|^3 \ \| \omega^{(n)}(t) \times \omega'(t) \|}, \tag{9}$$

$$K_2 = \frac{g(\omega''(t), \omega'(t) \times \omega^{(n)}(t)}{\| \omega'(t) \|^2 \| \omega'(t) \times \omega^{(n)}(t) \|}, \tag{10}$$

$$K_3 = \frac{g(\omega''(t), \omega'(t) \times \omega^{(n)}(t))\ g(\omega'(t), \omega^{(n)}(t))}{\| \omega'(t) \|^2 \| \omega'(t) \times \omega^{(n)}(t) \|^2}. \tag{11}$$

Proof. If $\omega(t)$ is a timelike curve, by differentiating the unit tangent vector T(t) in (4), and substituting into (5) and (8), we have

$$K_1 = \frac{g(T', D_2)}{u} = \frac{g(\omega''(t),\ (\omega'(t) \times \omega^{(n)}(t)) \times \omega'(t))}{\| \omega'(t) \|^3 \ \| \omega'(t) \times \omega^{(n)}(t) \|}.$$

By using the fact $(A \times B) \times C = A\ g(B, C) - B\ g(A, C)$, we get

$$K_1 = \frac{\| \omega'(t) \|^2 g(\omega''(t),\ \omega^{(n)}(t)) - \ g(\omega''(t),\ \omega'(t))\ g(\omega'(t),\ \omega^{(n)}(t))}{\| \omega'(t) \|^3 \ \| \omega'(t) \times \omega^{(n)}(t) \|}. \tag{12}$$

Since $g(A \times C,\ B \times D) = g(A, D)\ g(B, C) - g(A, B)\ g(C, D)$, we get

$$K_1 = \frac{g(\omega'(t) \times \omega''(t),\ \omega'(t) \times \omega^{(n)}(t)}{\| \omega'(t) \|^3 \ \| \omega'(t) \times \omega^{(n)}(t) \|}.$$

Similarly, we can obtain K_2 as follows:

$$K_2 = \frac{g(T', D_1)}{u} = \frac{g(\omega''(t), \omega'(t) \times \omega^{(n)}(t))}{\| \omega'(t) \|^2 \| \omega'(t) \times \omega^{(n)}(t) \|}.$$

Also, from (4) and (5), we can get K_3 as:

$$K_3 = \frac{g(D_2'(t),\ D_1(t))}{\| \omega'(t) \|} = \frac{g(D_1'(t) \times T(t),\ D_1(t))}{\| \omega'(t) \|}.$$

Similarly, we can find K_3 as:

$$K_3 = -\frac{g(D_1'(t),\ \omega'(t))\ g(\omega'(t),\ \omega^{(n)}(t)) -\ g(D_1'(t),\ \omega^{(n)}(t))\ g(\omega'(t),\ \omega'(t)}{\parallel \omega'(t) \parallel^2\ \ \parallel \omega'(t) \times \omega^{(n)}(t) \parallel}. \tag{13}$$

On the other hand, by differentiating D_1 in (4), we have

$$D_1' = \frac{(\omega''(t) \times \omega^{(n)}(t) + \omega'(t) \times\ \omega^{(n+1)}(t))}{\parallel \omega'(t) \times \omega^{(n)}(t) \parallel} - \frac{(\omega'(t) \times \omega^{(n)}(t))\frac{d}{dt} \parallel \omega'(t) \times \omega^{(n)}(t) \parallel}{\parallel \omega'(t) \times \omega^{(n)}(t) \parallel^2}. \tag{14}$$

Since, $(n+1)$-th derivative of the curve vanishes therefore by substituting (13) into (14), we deduce:

$$K_3 = \frac{g(\omega''(t), \omega'(t) \times \omega^{(n)}(t))\ g(\omega'(t), \omega^{(n)}(t))}{\parallel \omega'(t) \parallel^2 \parallel \omega'(t) \times \omega^{(n)}(t) \parallel^2}.$$

□

Corollary 3.2. *If $\omega(t)$ is a poly-timelike curve of degree n, the $n-th$ derivative of the curve can be expressed as a function of its base $\{T, D_1, D_2\}$ in the following form:*

$$\omega^n(t) = \parallel \omega^n(t) \parallel (\sinh\theta\, T(t) + \cosh\theta\, D_2(t)), \tag{15}$$

where $\theta = \int uK_1 dt$.

Theorem 3.3. *If $\omega(t)$ is a poly-timelike curve of degree n. The relation between curvatures K_1, K_2 and K_3 of the Flc-frame is obtained by:*

$$uK_1 = \frac{\left(\dfrac{K_3}{K_2}\right)'}{1 - \left(\dfrac{K_3}{K_2}\right)^2}. \tag{16}$$

Proof. If $\phi(t)$ is the Lorentzian angle between $D_2(t)$ and the $n-th$ derivative of the curve $\omega^n(t)$, then, we have

$$g(\omega^{(n)}(t), D_2(t)) = \parallel \omega^{(n)}(t) \parallel \cosh\phi.$$

From (4), we see that the vectors $\omega^{(n)}$, and D_1 are orthogonal and so,

$$g(\omega^{(n)}(t), T(t)) = - \parallel \omega^{(n)}(t) \parallel \sinh\phi,$$

which implies that,

$$g(\omega^{(n)}(t), \omega'(t)) = -u \parallel \omega^{(n)}(t) \parallel \sinh\phi. \tag{17}$$

Combining (10) and (11), we have

$$\frac{K_3}{K_2} = -\frac{\left[g\left(\omega'(t), \omega^{(n)}(t)\right)\right]}{\parallel \omega'(t) \times \omega^{(n)}(t) \parallel} = -\tanh\phi.$$

It follows that,

$$\phi = -\left[\tanh^{-1}\left(\frac{K_3}{K_2}\right)\right]. \tag{18}$$

Differentiating (18) yields:

$$\frac{d\phi}{dt} = \frac{-\left[\left(\frac{K_3}{K_2}\right)'\right]}{1-\left(\frac{K_3}{K_2}\right)^2}. \tag{19}$$

On the other hand, we have

$$\omega'(t) = uT(t), \qquad \omega''(t) = u'T(t) + u(K_1D_2(t) - K_3D_2(t)).$$

The inner product of $\omega'(t)$ and $\omega''(t)$ is given by:

$$g(\omega'(t), \omega''(t)) = -uu'. \tag{20}$$

Differentiating (17) yields:

$$g(\omega^{(n)}(t), \qquad \omega''(t)) = -u' \parallel \omega^{(n)}(t) \parallel \sin\phi - u \parallel \omega^{(n)}(t) \parallel \cos\phi \, \frac{d\phi}{dt}. \tag{21}$$

Substituting (17), (20) and (21) into (12), we obtain:

$$K_1 = \frac{1}{u}\frac{d\phi}{dt}. \tag{22}$$

Combining (19) and (22), we have the desired formula. $\square$

Example 3.1. *If ω is a poly-spatial curve of degree 9 given in Figure 1, by*

$$\omega(t) = \left(t^9 + t^5 + 7, t^7 - t^5 + 6, \frac{\sqrt{42}}{6}t^6\right).$$

Using (4), Flc- vector fields of $\omega(t)$ in Figures 2, 3 and 4 are as follows:

$$T = \left(\frac{9t^8+5t^4}{\sqrt{t^{10}\left(81t^6+41t^2+28\right)}}, \frac{7t^6-5t^4}{\sqrt{t^{10}\left(81t^6+41t^2+28\right)}}, \frac{\sqrt{42}t^5}{\sqrt{t^{10}\left(81t^6+41t^2+28\right)}}\right),$$

$$D_2 = \begin{pmatrix} -\dfrac{\sqrt{t^8\left(49t^4-28t^2+25\right)}}{\sqrt{t^{10}\left(81t^6+41t^2+28\right)}}, \\ -\dfrac{t^8\left(7t^2-5\right)\left(9t^4+5\right)}{\sqrt{t^8\left(49t^4-28t^2+25\right)}\sqrt{t^{10}\left(81t^6+41t^2+28\right)}}, \\ -\dfrac{\sqrt{42}t^9\left(9t^4+5\right)}{\sqrt{t^8\left(49t^4-28t^2+25\right)}\sqrt{t^{10}\left(81t^6+41t^2+28\right)}} \end{pmatrix},$$

$$D_1 = \left(0, \frac{\sqrt{42}t^5}{\sqrt{t^8\left(49t^4-28t^2+25\right)}}, \frac{5t^4-7t^6}{\sqrt{t^8\left(49t^4-28t^2+25\right)}}\right).$$

The Flc-curvatures of $\omega(t)$ are defined by:

$$K_1 = \frac{2t^{13}\left(441t^6 - 378t^4 + 205t^2 + 70\right)}{\sqrt{t^8\left(49t^4 - 28t^2 + 25\right)}\left(t^{10}\left(81t^6 + 41t^2 + 28\right)\right)^{3/2}}, \tag{23}$$

$$K_2 = \frac{\sqrt{42}\left(7t^2 + 5\right)}{t^2\sqrt{t^8\left(49t^4 - 28t^2 + 25\right)}\left(81t^6 + 41t^2 + 28\right)}, \tag{24}$$

$$K_3 = -\frac{\sqrt{42}\left(63t^6 + 45t^4 + 35t^2 + 25\right)}{t^6\left(49t^4 - 28t^2 + 25\right)\left(81t^6 + 41t^2 + 28\right)}. \tag{25}$$

Also by using (16), *we get*

$$uk_1 = \frac{\dfrac{d}{dt}\left(\dfrac{K_3}{K_2}\right)}{1 - \left(\dfrac{K_3}{K_2}\right)^2} = \frac{2t^3(70 + 205t^2 - 378t^4 + 441t^6)}{\sqrt{t^8(25 - 28t^2 + 49t^4)}(28 + 41t^2 + 81t^6)}.$$

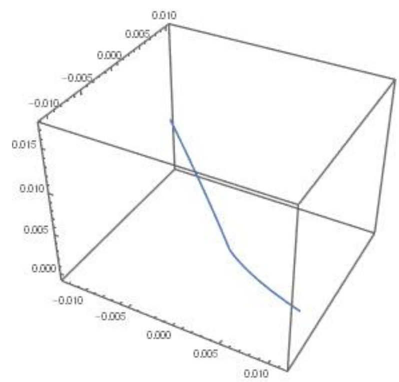

Figure 1: The timelike curve $\gamma(t)$.

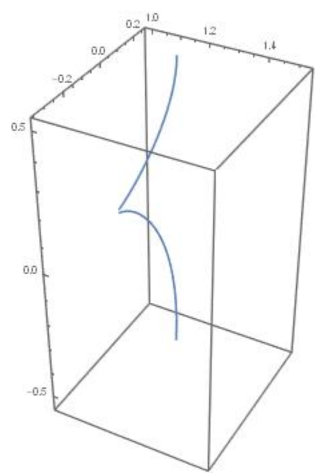

Figure 2: The tangent vector $T(t)$.

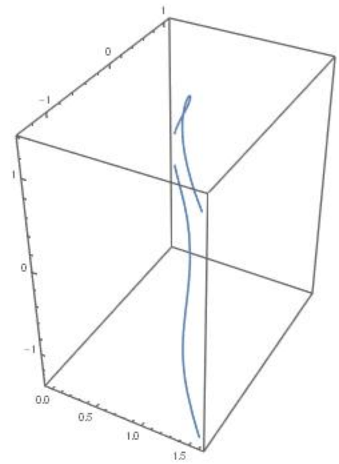

Figure 3: The Flc-normal vector $D_2(t)$.

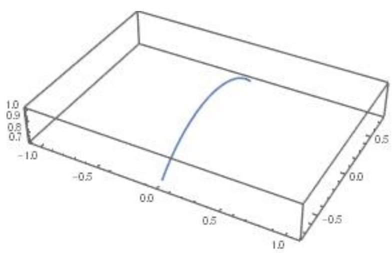

Figure 4: The Flc-binormal vector $D_1(t)$.

4 Integral Curves

Integral curves can be obtained by integration of the normal vector or the binormal vector of another curve [5]. We introduce a new type of integral curve in which the Frenet vectors, curvature, and torsion for these curves are generated by using the Flc-frame and the Flc-curvatures.

Definition 4.1. *If $\omega(t)$ is a timelike curve. The integral curve $\gamma(t)$ of $D_2(t)$ is defined as:*

$$\gamma(t) = \int D_2(t)dt. \tag{26}$$

Theorem 4.1. *If $\omega(t)$ is a timelike curve and $\gamma(t)$ is the integral curve of $D_2(t)$ with $K_1, K_3 \neq 0$, then the Frenet vector fields are:*

$$(T(t))_\gamma(t) = ((D_2(t))_\omega)_\omega,$$

$$(N(t))_\gamma(t) = \frac{K_1}{\sqrt{|K_3^2 - K_1^2|}}(T(t))_\omega - \frac{K_3}{\sqrt{|K_3^2 - K_1^2|}}((D_1(t))_\omega)_\omega,$$

$$(B(t))_\gamma(t) = -\frac{K_3}{\sqrt{|K_3^2 - K_1^2|}}(T(t))_\omega - \frac{K_1}{\sqrt{|K_3^2 - K_1^2|}}((D_1(t))_\omega)_\omega.$$

Proof. If $\omega(t)$ is a timelike curve and $\gamma(t)$ be the integral curve of $(D_2(t))_\omega$ with $K_1, K_3 \neq 0$. By using (5) and (26) and taking the derivatives, we get

$$\gamma'(t) = (D_2(t))_\omega, \tag{27}$$

$$\gamma''(t) = D_2'(t) = u(K_1 T(t) + K_3(D_1(t))_\omega). \tag{28}$$

Taking the cross-product of (27) and (28), we get

$$\gamma'(t) \times \gamma''(t) = u(-K_3 T(t) - K_1(D_1(t))_\omega). \tag{29}$$

Applying Frenet-Equations (3), we have the desired formulae. □

Corollary 4.1. *If $\omega(t)$ is a timelike curve and $\gamma(t)$ is integral curve of $(D_2(t))_\omega$ of a timelike poly-curve $\omega(t)$ with $K_1, K_3 \neq 0$, then $\gamma(t)$ is a spacelike curve in which Frenet-normal $N_\gamma(t)$ is a spacelike vector if $-K_1 < K_3 < K_1$ and $\gamma(t)$ is a spacelike curve in which Frenet-normal $N_\gamma(t)$ is a timelike vector if $K_3 > K_1$ or $K_3 < -K_1$.*

Theorem 4.2. *If $\omega(t)$ is a timelike curve and $\gamma(t)$ is the integral curve of $(D_2(t))_\omega$ of a poly-timelike curve $\omega(t)$ with $K_1,\ K_3 \neq 0$, then the Frenet curvature and Frenet torsion of $\gamma(t)$ are respectively given by:*

$$(\kappa(t))_\gamma = u\sqrt{|K_3^2 - K_1^2|},$$

$$(\tau(t))_\gamma = \frac{K_1^2 \dfrac{d}{dt}\left(\dfrac{K_3}{K_1}\right) - uK_2(K_3^2 - K_1^2)}{|K_3^2 - K_1^2|}.$$

Proof. If $\omega(t)$ is a spacelike curve with a spacelike Flc-normal $(D_2(t))_\omega$ vector and $\gamma(t)$ is the integral curve of $(D_2(t))_\omega$ of a poly-spacelike curve $\omega(t)$ with $K_1,\ K_3 \neq 0$. By taking the derivative of (26), we get

$$\gamma'''(t) = \left(u'K_1 + uK_1' + u^2K_2K_3\right)T(t) + u^2\left(K_3^2 - K_1^2\right)(D_2(t))_\omega + \left(u'K_3 + uK_3' + u^2K_1K_2\right)(D_1(t))_\omega.$$

By using (3), (29) and (27), the Frenet curvature and Frenet torsion of $\gamma(t)$ are given by:

$$(\kappa(t))_\gamma = \frac{\parallel \gamma'(t) \times \gamma''(t) \parallel}{\parallel \gamma'(t) \parallel^3} = u\sqrt{|K_3^2 - K_1^2|},$$

$$(\tau(t))_\gamma = \frac{g(\gamma'''(t), \gamma'(t) \times \gamma''(t))}{\parallel \gamma'(t) \times \gamma''(t) \parallel^2} = \frac{K_1^2 \dfrac{d}{dt}\left(\dfrac{\omega K_3}{K_1}\right) + uK_2(K_3^2 - K_1^2)}{|K_3^2 - K_1^2|}.$$

□

Definition 4.2. *If $\omega(t)$ is a poly-timelike curve, the integral curve $\eta(t)$ of $(D_1(t))_\omega$ is defined as:*

$$\eta(t) = \int (D_1(t))_\omega dt. \tag{30}$$

Theorem 4.3. *If $\omega(t)$ is a spacelike curve in which Flc-binormal $(D_1(t))_\omega$ is a spacelike vector and $\gamma(t)$ be the integral curve of $(D_1(t))_\omega$ with K_2, $K_3 \neq 0$, then the Frenet vector fields are:*

$$(T(t))_\eta = (D_1(t))_\omega,$$

$$(N(t))_\eta = \frac{-K_2}{\sqrt{|K_3^2 - K_2^2|}}(T(t))_\omega + \frac{K_3}{\sqrt{|K_3^2 - K_2^2|}}(D_2(t))_\omega,$$

$$(B(t))_\eta = \frac{-K_3}{\sqrt{|K_3^2 - K_2^2|}}(T(t))_\omega + \frac{K_2}{\sqrt{|K_3^2 - K_2^2|}}(D_2(t))_\omega.$$

Corollary 4.2. *If $\omega(t)$ is a timelike curve and $\eta(t)$ is integral curve of $(D_1(t))_\omega$ of a poly-timelike curve $\omega(t)$ with $K_2, K_3 \neq 0$, then $\eta(t)$ is a spacelike curve in which Frenet-normal $N_\eta(t)$ is a spacelike vector if $-K_2 < K_3 < K_2$ and $\eta(t)$ is a spacelike curve in which Frenet-normal $N_\eta(t)$ is a T-like vector if $K_3 > K_2$ or $K_3 < -K_2$.*

Theorem 4.4. *If $\omega(t)$ is a spacelike curve in which Flc-binormal $(D_1(t))_\omega$ is a spacelike vector and $\eta(t)$ is the integral curve of $(D_1(t))_\omega$ of a poly-spacelike curve $\omega(t)$ with K_2, $K_3 \neq 0$, then the Frenet curvature and Frenet torsion of $\eta(t)$ are respectively given by:*

$$(\kappa(t))_\eta = u\sqrt{|K_3^2 - K_2^2|},$$

$$(\tau(t))_\eta = \frac{K_2^2 \dfrac{d}{dt}\left(\dfrac{K_3}{K_2}\right) + uK_1(K_3^2 - K_2^2)}{|K_3^2 - K_2^2|}.$$

5 Conclusion

This research extends the application of the Flc-frame to poly-timelike curves in Minkowski 3-space, overcoming the limitations of the traditional Frenet frame, especially at points where the second derivative of the curve becomes zero. By introducing the Flc-frame and Flc-equations for timelike curves, we derived the Flc-curvatures K_1, K_2 and K_3, and established explicit relationships among them. These curvatures offer a thorough description of the geometric properties of timelike curves, even in cases where the Frenet frame is undefined. A key contribution is the development of new integral curves generated from the Flc-frame. These integral curves, obtained by integrating the Flc-normal and Flc-binormal vectors, enable the explicit computation of Frenet vectors, curvature, and torsion in terms of the Flc-curvatures.

The detailed example presented in this paper demonstrates the practical application of the Flc-frame, showcasing its effectiveness in analyzing the geometric properties of poly-timelike curves. The results emphasize the advantages of the Flc-frame over traditional methods, offering a versatile tool for both theoretical and practical applications in differential geometry and mathematical physics.

Moreover, the relationships between the Flc-curvatures and the derived integral curves provide new insights into the geometric structure of timelike curves in Minkowski space. The ability to compute the Frenet apparatus directly from the Flc-frame and Flc-curvatures simplifies the analysis of complex curves and deepens the understanding of their behavior. Future work includes adapting the Flc-frame to null curves and higher-dimensional spaces.

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References

[1] R. L. Bishop (1975). There is more than one way to frame a curve. *The American Mathematical Monthly*, *82*(3), 246–251. http://doi.org/10.2307/2319846.

[2] S. Büyükkütük, I. Kisi & G. Öztürk (2018). A characterization of non-lightlike curves with respect to parallel transport frame in Minkowski space-time. *Malaysian Journal of Mathematical Sciences*, *12*(2), 223–234.

[3] M. Dede (2023). On polynomial space curves with Flc-frame. *Turkish Journal of Mathematics and Computer Science*, *15*(2), 414–422. https://doi.org/10.47000/tjmcs.1127766.

[4] A. Elsharkawy (2024). Exploring Hasimoto surfaces within equiform geometry in Minkowski space. *Physica Scripta*, *100*(1), Article ID: 016101. https://doi.org/10.1088/1402-4896/ad954e.

[5] A. Elsharkawy & H. Baizeed (2025). Some integral curves according to quasi-frame in Euclidean 3-space. *Scientific African*, *27*, Article ID: e02583. https://doi.org/10.1016/j.sciaf.2025.e02583.

[6] K. Eren, K. H. Ayvaci & S. Senyurt (2022). On characterizations of spherical curves using Frenet like curve frame. *Honam Mathematical Journal*, *44*(3), 391–401. https://doi.org/10.5831/HMJ.2022.44.3.391.

[7] E. Hamouda, C. Cesarano, S. Askar & A. Elsharkawy (2021). Resolutions of the jerk and snap vectors for a quasi curve in Euclidean 3-space. *Mathematics*, *9*(23), Article ID: 3128. https://doi.org/10.3390/math9233128.

[8] Y. Li, K. Eren, K. H. Ayvacı & S. Ersoy (2022). Simultaneous characterizations of partner ruled surfaces using Flc frame. *AIMS Mathematics*, *7*(11), 20213–20229. https://doi.org/10.3934/math.20221106.

[9] Y. Li, K. Eren, S. Ersoy & A. Savić (2024). Modified sweeping surfaces in Euclidean 3-space. *Axioms*, *13*(11), Article ID: 800. https://doi.org/10.3390/axioms13110800.

[10] Y. Li, J. Li, Z. Yang, R. Abdel Baky & M. Saad (2024). Directional developable surfaces and their singularities in Euclidean 3-space. *Filomat*, *38*(32), 11333–11347. http://dx.doi.org/10.22541/au.168236284.46863569/v1.

[11] I. Mohd & Y. Dasril (2019). Formulate a relationship between saddle points on surfaces and inflection points on curves. *Malaysian Journal of Mathematical Sciences*, *13*(3), 439–446.

[12] M. Raussen (2008). *Elementary Differential Geometry: Curves and Surfaces*. Aalborg University, Denmark.

[13] S. Şenyurt, K. Eren & K. H. Ayvacı (2022). A study on inextensible flows of polynomial curves with Flc frame. *Applications & Applied Mathematics*, *17*(1), 123–133.

[14] S. Şenyurt, K. Eren & K. H. Ayvacı (2024). Characterizations of Tzitzeica curves using Flc frame. *Sigma Journal of Engineering and Natural Sciences*, *42*(1), 37–41. https://doi.org/10.14744/sigma.2024.00005.

[15] S. Yılmaz & M. Turgut (2010). A new version of Bishop frame and an application to spherical images. *Journal of Mathematical Analysis and Applications*, *371*(2), 764–776. https://doi.org/10.1016/j.jmaa.2010.06.012.

[16] Y. Zhu, Y. Li, K. Eren & S. Ersoy (2025). Sweeping surfaces of polynomial curves in Euclidean 3-space. *Analele Stiintifice ale Universitatii Ovidius Constanta: Seria Matematica*, *33*(1), 293–311. https://doi.org/10.2478/auom-2025-0015.